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LETTER TO THE EDITOR

New classes of conserved quantities associated with non-Noether symmetries

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Abstract. A recent result of Hojman and Harleston has been used to extend the known relationship between conserved quantities and non-Noether symmetries. We show that N constants of the motion may be derived from each non-Noether symmetry of an N-dimensional Lagrangian system; this applies whether the symmetry is continuous or discrete. In addition, the properties of the Cartan form are utilised to give a new proof of the Hojman–Harleston theorem.

A non-Noether symmetry in Lagrangian mechanics is defined by the fact that it transforms the solution manifold into itself, but does not leave the action integral invariant; examples have been given by Lutzky (1978) and Prince and Eliezer (1980, 1981). An important property is that each continuous non-Noether symmetry gives rise to a constant of the motion, which may be constructed from the symmetry generator without further integration (Lutzky 1979a, b). It has, of course, been known for a long time that constants of the motion may be associated with Noether symmetries (Noether 1918); however, the considerations relating conserved quantities to non-Noether symmetries are of a somewhat different nature. For example, in contrast to the Noether case, *discrete* non-Noether symmetries can sometimes lead to conserved quantities for Lagrangian systems (Lutzky 1981).

A basic tool used in studying non-Noether symmetries has been the following theorem (Lutzky 1979b, 1981): let $L(q, \dot{q}, t)$ and $\tilde{L}(q, \dot{q}, t)$ be distinct Lagrangians leading to the same equations of motion, and let V^l_k be defined by the relation $\partial^2\tilde{L}/\partial\dot{q}_k\partial\dot{q}_i = V^l_k\, \partial^2 L/\partial\dot{q}_l\, \partial\dot{q}_i$; then the quantity $\det\{V^l_k\}$ is a constant of the motion (note that V^l_k is well defined if $\det\{\partial^2 L/\partial\dot{q}_l\, \partial\dot{q}_i\} \neq 0$). In a recent paper, Hojman and Harleston (1981) have proved a more general theorem which states that not only the determinant but *all* of the invariants of the matrix V^l_k are constants of the motion. Here we shall explore the consequences of this generalisation for the theory of non-Noether integrals, and in particular we shall display the resulting *class* of conserved quantities which may be obtained from a continuous non-Noether symmetry. We shall also show how a discrete non-Noether symmetry gives rise to a class of conserved quantities. Finally, we present an alternative approach to the theorem of Hojman and Harleston, originally proved by these authors through manipulation of the Euler equations. Our derivation relies on the properties of the Cartan form, and is in the spirit of recent treatments of Lagrangian systems employing the tools of modern differential geometry (see, for instance, Crampin 1977, Sarlet and Cantrijn 1981a, b).

We begin by showing how the Hojman–Harleston theorem enables us to construct a class of conserved quantities from each continuous non-Noether symmetry. Let the Lagrangian $L(q, \dot{q}, t)$ lead to the equations of motion

$$\ddot{q}^l_1 = \alpha^l(q, \dot{q}, t) \qquad l = 1, 2, \ldots, N; \tag{1}$$

we further define the Cartan form

$$\theta = \left(L - \frac{\partial L}{\partial \dot{q}^l}\dot{q}^l\right) \mathrm{d}t + \frac{\partial L}{\partial \dot{q}^l}\mathrm{d}q^l \tag{2}$$

and the vector field

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^l\frac{\partial}{\partial q^l} + \alpha^l\frac{\partial}{\partial \dot{q}^l} \tag{3}$$

which represents the total time derivative along trajectories of (1). (Thus, if ϕ is a function, $\Gamma(\phi) = \dot{\phi}$.)

It is well known (Sarlet and Cantrijn 1981a, b) that the operator $G = \xi(q, t)\, \partial/\partial t + \eta^l(q, t)\, \partial/\partial q^l$ generates a point symmetry of (1) whenever

$$[E, \Gamma] = -\dot{\xi}\Gamma \tag{4}$$

where $E = G + (\dot{\eta}^l - \dot{\xi}\dot{q}^l)\, \partial/\partial \dot{q}^l$, and Γ is the vector field given by (3). Furthermore, if θ is the Cartan form (2), the equation

$$i_\Gamma\, \mathrm{d}\theta = 0 \tag{5}$$

is equivalent to the equations of motion (1). Defining $\theta' = \mathscr{L}_E\theta$, it follows by direct computation that

$$\theta' = \left(L' - \dot{q}^l\frac{\partial L'}{\partial \dot{q}^l}\right) \mathrm{d}t + \frac{\partial L'}{\partial \dot{q}^l}\mathrm{d}q^l$$

where $L' = E\{L\} + \dot{\xi}L$; thus θ' is again a Cartan form, for the Lagrangian L'. We may also show that if E satisfies equation (4), and if (5) holds, then we have $i_\Gamma\, \mathrm{d}\theta' = 0$; for the proof we put $i_\Gamma\, \mathrm{d}\theta' = i_\Gamma\mathscr{L}_E\, \mathrm{d}\theta = \mathscr{L}_E i_\Gamma\, \mathrm{d}\theta - i_{[E,\Gamma]}\, \mathrm{d}\theta = \dot{\xi}i_\Gamma\, \mathrm{d}\theta = 0$. Consequently L' and L both lead to the same equations of motion (1), and by the Hojman–Harleston theorem we may conclude that the matrix Δ, defined by $\partial^2 L'/\partial\dot{q}^m\, \partial\dot{q}^p = \Delta^l_m\, \partial^2 L/\partial\dot{q}^l\, \partial\dot{q}^p$, has the property that *all* of its invariants are constants of the motion. It is straightforward to obtain an explicit expression for Δ^l_m; for this purpose we establish the convenient notation $L_{lp} = \partial^2 L/\partial\dot{q}^l\, \partial\dot{q}^p$, with inverse matrix L^{kl}, so that $L^{kl}L_{lp} = \delta^k_p$. The equation defining Δ^l_m may be written $L'_{mp} = \Delta^l_m L_{lp}$, and multiplying by L^{pk} gives $\Delta^k_m = L'_{mp}L^{pk}$. Recalling that $L' = E\{L\} + \dot{\xi}L$, we may carry out the differentiations in L'_{mp} to obtain, after some algebra, the result

$$\Delta^k_m = -\dot{\xi}\delta^k_m + B^k_m + L^{kp}B^l_p L_{lm} + L^{kp}E\{L_{pm}\} \tag{6}$$

where

$$B^k_m = \frac{\partial\eta^k}{\partial q^m} - \dot{q}^k\frac{\partial\xi}{\partial q^m}.$$

The simplest invariant to calculate is the trace, which takes the form

$$\Delta^k_k = -N\dot{\xi} + 2B^k_k + E\{\ln D\}$$

where $D = \det\{L_{lm}\}$. This quantity has already been recognised as a constant of the motion obtainable from a non-Noether symmetry (Lutzky 1979a, b); however, the remaining $N-1$ constants of the motion, corresponding to the remaining $N-1$ invariants of the matrix Δ, have not previously been noted. Since any function of invariants is itself an invariant, we see that the coefficients of the characteristic equation of Δ, the eigenvalues of Δ, and the quantities $\mathrm{Tr}\{\Delta^k\}$, $k = 1, 2, \ldots, N$, are all conserved quantities. At most N of these quantities can be functionally independent, and in particular cases the total number of functionally independent invariants may be less than N.

An instructive example is provided by the N-dimensional harmonic oscillator, with Lagrangian $L = \frac{1}{2}(\dot{q}_l\dot{q}_l - q_lq_l)$, for which $L_{lm} = \delta_{lm}$. In this case conserved quantities may be found as invariants of the matrix whose elements are L'_{mn}, $L' = E\{L\} + \dot{\xi}L$. Generators of non-Noether symmetries for the harmonic oscillator are obtainable from the work of Prince and Eliezer (1980), and a particular one may be specified by (for example) $\xi = q_1 \sin t$, $\eta_k = q_kq_1 \cos t$. These functions determine a symmetry E, and therefore also a matrix L'_{mn}; evaluating the invariants of this matrix then leads to the N functionally independent constants of the motion $q_k \cos t - \dot{q}_k \sin t$, $k = 1, 2, \ldots, N$.

There are, of course, simpler methods of treating the harmonic oscillator; the value of this example is that it encourages the hope that there may exist more complicated systems for which N functionally independent integrals may be found from the knowledge of a single non-Noether symmetry.

To emphasise the special properties of non-Noether symmetries, we point out that (4) is satisfied by all point symmetries of Γ; however, if a symmetry is Noether, it must satisfy the additional requirement $\mathscr{L}_E\, \mathrm{d}\theta = 0$ (Crampin 1977, Sarlet and Cantrijn 1981a, b), which leads to the condition $L' = E\{L\} + \dot{\xi}L = \dot{f}$, $f = f(q, t)$. (Another approach to this condition has been given by Lutzky (1978).) Thus, for a Noether symmetry we have $L'_{ml} = 0$ and $\Delta^k_l = 0$; this explains why only non-Noether symmetries can lead to non-trivial conserved quantities through the procedures described above.

The above development is valid for continuous symmetries; we now comment briefly on the case of discrete, non-Noether symmetries. Let the trajectories $Q^l = Q^l(T)$ in Q, T space be related to the trajectories $q^l = q^l(t)$ in q, t space by the discrete transformation $q^l = q^l(Q, T)$, $t = t(Q, T)$. It can be shown that if the trajectories in q, t space are solution curves of the dynamical system with Lagrangian $L(q, \dot{q}, t)$, then the curves in Q, T space are solution curves for the Lagrangian

$$\tilde{L}(Q, Q', T) = L(q\{Q, T\}, \dot{q}\{Q, Q', T\}, t\{Q, T\})\left(\frac{\partial t}{\partial T} + \frac{\partial t}{\partial Q^k} Q'^k\right) \tag{7}$$

where $Q' = \mathrm{d}Q/\mathrm{d}T$. If the discrete transformation leaves the equations of motion form invariant, then it follows that $\tilde{L}(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$ both give rise to the same equations of motion. We may accordingly define the matrix $\Delta^k_m = \tilde{L}_{mp}L^{pk}$, whose invariants are constants of the motion (by the Hojman–Harleston theorem). Until now, only $\det\{\Delta^k_m\}$ was known to be conserved for the case of a discrete non-Noether symmetry (Lutzky 1981).

A general condition which a point symmetry of a Lagrangian system must satisfy in order to be Noetherian is given by

$$\tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \dot{F}$$

where $F = F(q, t)$, and the functional form of $\tilde{L}$ is specified by (7). This criterion is applicable whether the transformation is continuous or discrete; thus the previously given condition for a continuous Noether symmetry is derivable from this one (Lutzky 1978). It follows that for a discrete Noether symmetry we have $\Delta_l^k = \delta_l^k$, and we may therefore conclude that the application of the Hojman–Harleston theorem to discrete transformations can lead to non-trivial conserved quantities only if the symmetry is non-Noether.

Finally, we give an alternative proof of the Hojman–Harleston theorem, using the calculus of differential forms. Let the Lagrangians L and $\tilde{L}$ both lead to the same equations of motion (1), and let θ and $\tilde{\theta}$ be the corresponding Cartan forms. We have $\mathscr{L}_\Gamma \, d\theta = d(i_\Gamma \, d\theta)$ and $\mathscr{L}_\Gamma \, d\tilde{\theta} = d(i_\Gamma \, d\tilde{\theta})$; from equation (5) and its analogue for $\tilde{\theta}$ we then obtain the results

$$\mathscr{L}_\Gamma \, d\theta = 0 \qquad \mathscr{L}_\Gamma \, d\tilde{\theta} = 0. \tag{8}$$

Construct the $N+1$ volume forms $\Omega_k = dt \wedge d\tilde{\theta} \wedge d\tilde{\theta} \wedge \ldots \wedge d\theta \wedge d\theta$, $k = 0, 1, 2, \ldots, N$, where Ω_k contains k factors of $d\tilde{\theta}$ and $N-k$ factors of $d\theta$. It follows from (8) that $\mathscr{L}_\Gamma \Omega_k = 0$ for all k. Each Ω_k may be expressed in the form $\Omega_k = \rho_k \, dt \wedge dq^1 \wedge \ldots \wedge dq^N \wedge d\dot{q}^1 \wedge \ldots \wedge d\dot{q}^N$, where the ρ_k are zero forms (functions), and it is easily seen that the ratio of any two of these functions is a constant of the motion. For example, let $\Omega_l = \sigma\Omega_m$, and calculate $\mathscr{L}_\Gamma \Omega_l = 0 = \Omega_m \mathscr{L}_\Gamma \sigma$. Then $\mathscr{L}_\Gamma \sigma = \Gamma\{\sigma\} = \dot{\sigma} = 0$, and σ is conserved; furthermore, from the definition of Ω_l and Ω_m we must have $\sigma = \rho_l/\rho_m$. A specific expression for ρ_k may be derived by expanding Ω_k, with $d\theta$ and $d\tilde{\theta}$ expressed in terms of coordinate differentials. We obtain

$$\begin{aligned}
\rho_k &\sim \varepsilon^{i_1 \cdots i_N} \varepsilon^{j_1 \cdots j_N} \tilde{L}_{i_1 j_1} \ldots \tilde{L}_{i_k j_k} L_{i_{k+1} j_{k+1}} \ldots L_{i_N j_N} \\
&= (\Delta_{i_1}^{\alpha_1} \ldots \Delta_{i_k}^{\alpha_k})(\varepsilon^{i_1 \cdots i_N} \varepsilon^{j_1 \cdots j_N})(L_{\alpha_1 j_1} \ldots L_{\alpha_k j_k} L_{i_{k+1} j_{k+1}} \ldots L_{i_N j_N}) \\
&= (\Delta_{i_1}^{\alpha_1} \ldots \Delta_{i_k}^{\alpha_k})(\delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^{i_1 i_2 \ldots i_k})(N-k)!\, D
\end{aligned}$$

where $D = \det\{L_{ij}\}$ and the symbol $\sim$ means equality to within a multiplicative constant (whose exact value is unimportant for our purposes). It can be shown (Lovelock and Rund 1975) that

$$k!\, \Delta_{(k)} = (\Delta_{i_1}^{\alpha_1} \ldots \Delta_{i_k}^{\alpha_k})(\delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^{i_1 i_2 \ldots i_k})$$

where $\Delta_{(k)}$ is the sum of the $k \times k$ principal minors of Δ; and since $\rho_0 \sim D$, we see that each conserved quantity ρ_k/ρ_0 is proportional to $\Delta_{(k)}$. Then the $\Delta_{(k)}$ are themselves conserved quantities, and since the $\Delta_{(k)}$ constitute a set of invariants of the matrix Δ (being proportional to the coefficients of the characteristic equation) the theorem is proved.

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